STACK STRUCTURES ON GIT QUOTIENTS PARAMETRIZING HYPERSURFACES

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ABSTRACT. We suggest to endow Mumford's GIT quotient scheme with a stack structure, by replacing $\mathrm{Proj}(-)$ of the invariant ring with its stack theoretic analogue. We analyse the stacks resulting in this way from classically studied invariant rings, and in particular for binary forms of low degree. Our viewpoint is that the stack structure carries interesting geometric information that is intrinsically present in the invariant ring, but lost when passing to its $\mathrm{Proj}(-)$.

1. INTRODUCTION

Let G be a reductive group acting on $\mathbf{P}^N$ via a linear representation, and let $Y \subseteq \mathbf{P}^N$ be a G-invariant subscheme with homogeneous coordinate ring S. Thus we consider a linearized action of G on $Y = \mathrm{Proj}(S)$. Let $R = S^G$ be the invariant ring. According to Mumford's geometric invariant theory, the semistable locus Y^{ss} admits a good quotient, which is the projective scheme

$$X = \mathrm{Proj}(R).$$

In classical invariant theory, a central question was to find explicit presentations for the invariant ring R in specific examples. Such presentations give explicit equations for the GIT quotient scheme X.

Let $0 \in \mathrm{Spec}(R)$ be the vertex, defined by the ideal R_+ generated by elements of strictly positive degree. Then $\mathbf{G}_m$ acts on the complement $\mathrm{Spec}(R) \smallsetminus \{0\}$, and the quotient scheme is $\mathrm{Proj}(R)$. The $\mathbf{G}_m$-action is free if R is generated in degree 1, but not in general. The invariant rings we will consider are not generated in degree 1, and thus it is natural to consider also the stack quotient $\mathscr{X}$ of the same action of $\mathbf{G}_m$ on $\mathrm{Spec}(R) \smallsetminus \{0\}$. This stack will be called the *stacky GIT quotient*.

Thus the stacky GIT quotient $\mathscr{X}$ is a Deligne-Mumford stack with the usual projective GIT quotient scheme X as underlying coarse space. In the language of Alper's stack theoretic treatment of GIT [3], the scheme X is a "good moduli space" for the quotient stack $[Y^{\mathrm{ss}}/G]$, and as the natural map $Y^{\mathrm{ss}} \to \mathscr{X}$ is G-invariant (in the 2-categorical sense), the quotient map from $[Y^{\mathrm{ss}}/G]$ to its good moduli space factors through the stacky GIT quotient:

$$[Y^{\mathrm{ss}}/G] \to \mathscr{X} \to X$$

Thus the stacky GIT quotient sits somewhere between the full stack quotient and the GIT quotient scheme. It has richer structure than the latter, but is simpler than the full stack quotient, which is not Deligne-Mumford in general. On the other hand it is unclear exactly

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what the stacky enrichment of the usual GIT quotient captures, and we do not know of any sense in which it is a *quotient*, for instance in terms of a universal property.

Our aim is to analyse the relationship between the stacky GIT quotient and the GIT quotient scheme in the examples studied in classical invariant theory, where explicit presentations for the invariant ring is known. Thus we are concerned with the action of $G = \mathrm{SL}(n+1)$ by substitution on the projective space Y of degree d hypersurfaces in $\mathbf{P}^n$ for n and d small. More precisely, we consider the actions of $\mathrm{SL}(2)$ on binary quartics, quintics and sextics; of $\mathrm{SL}(3)$ on cubic curves; and of $\mathrm{SL}(4)$ on cubic surfaces.

The invariant ring of binary quartics and cubic curves are just weighted polynomial rings in two variables. The invariant rings of binary quintics, binary sextics, and cubic surfaces are more interesting, and admits a special presentation (see Equation 5.1) involving a certain polynomial F. We find that for a ring R of this form, the corresponding stack $\mathscr{X}$ can essentially (precisely, up to an essentially trivial $\boldsymbol{\mu}_r$-gerbe) be reconstructed from its coarse space $X = \mathrm{Proj}(R)$ together with the divisor $Z(F)$ on it. On the other hand, this divisor cannot be described in terms of the intrinsic geometry of the GIT quotient *scheme*. Thus, for invariant rings of the type (5.1), the divisor $Z(F)$ is the essential piece of information that is intrinsically present in the ring, and remembered by the stacky GIT quotient, but "forgotten" by the GIT quotient scheme.

Moreover, in the case of $\mathrm{SL}(2)$ acting on binary forms of degree up to 6, we consider the classification of binary forms according to symmetry, i.e. their stabilizer groups in $\mathrm{SL}(2)$: Binary forms with prescribed symmetry group correspond to locally closed loci in the GIT quotient scheme X. We observe that the loci of binary forms with extra symmetries (i.e. larger symmetry group than the generic one) occur as

(1) singular points of X,
(2) the divisor $Z(F)$,
(3) singular points of $Z(F)$, or
(4) singular points of the singular locus of $Z(F)$.

As the stacky GIT quotient $\mathscr{X}$ remembers the divisor $Z(F)$, this enables us to describe the loci of binary forms with extra symmetries in terms of the intrinsic geometry of $\mathscr{X}$. We remark that the same statement trivially holds for the stack quotient $[Y^{\mathrm{ss}}/G]$, but is not obvious for the stacky GIT quotient, since the automorphism groups of its points do not coincide with the stabilizer groups for the $\mathrm{SL}(2)$ action.

The approach in this text is entirely dependant on the invariant ring having the special presentation (5.1). This structure is very special, although it is typical for the invariant rings determined explicitly by the invariant theorists of the 19th century. Already for binary forms of degree larger than 6, the present approach does not apply: The locus of binary forms with extra symmetries has codimension at least 2 as soon as the degree exceeds 6, and hence does not contain a divisor $Z(F)$. We remark that the invariant ring for binary forms of degree 8 has been explicitly described by Shioda [15], and its structure is indeed more complicated than (5.1). Beyond those examples treated here, the only cases known to the author that can be studied with similar methods are the actions of finite subgroups of $\mathrm{SL}(2)$ on $\mathbf{P}^1$ (with the natural linearization given by the action of $\mathrm{SL}(2)$ on $\mathbf{A}^2$), whose invariant rings have a structure close to that of (5.1) [16, Section 4.5].

The text roughly consists of two parts: In Sections 3, 4 and 5 we recall standard stack theoretic notions (the root construction, rigidification, the canonical smooth stack), and investigate their meaning for the stacks arising from graded rings of the form (5.1). In Sections 6, 7 and 8 we analyse the stacky GIT quotients corresponding to the classically studied actions of $\mathrm{SL}(n+1)$ on hypersurfaces in $\mathbf{P}^n$. The material in this second part has a

classic taste, and is undoubtedly well known. I claim originality only for the interpretation of these results in terms of the stacky GIT quotient. It should also be remarked that the stacky GIT quotient for binary sextics, and its "memory" of $Z(F)$, has been considered by Hassett [11, Section 3.1].

For an overview of classical invariant theory, and for more detailed references to original works than is given here, the reader is referred to the book by Dolgachev [7], which has been very useful in preparing this text. I learnt the right language (the root construction, etc.) for these investigations from a talk on toric stacks by B. Fantechi at the Institut Mittag-Leffler in May, 2007.

2. Notation

We work over an algebraically closed field k of characteristic zero. Following Fantechi et. al. [9], we define a *DM stack* to be a separated Deligne-Mumford stack.

Let $R = \bigoplus_{d\geq 0} R_d$ be a nonnegatively graded k-algebra with $R_0 = k$, and let R_+ be the maximal ideal generated by elements of strictly positive degree. Thus $\operatorname{Spec}(R)$ is a cone, with vertex $0 \in \operatorname{Spec}(R)$ defined by R_+. We write

$$\mathscr{P}roj(R) = [(\operatorname{Spec}(R) \smallsetminus \{0\}))/\mathbf{G}_m] \tag{2.1}$$

for the stack quotient by the natural action of $\mathbf{G}_m$ associated to the grading. The coarse space of $\mathscr{X} = \mathscr{P}roj(R)$ is the usual scheme $X = \operatorname{Proj}(R)$. Note that line bundles on $\mathscr{X}$ can be identified with $\mathbf{G}_m$-linearized line bundles on $\operatorname{Spec}(R) \smallsetminus \{0\}$. Thus, the graded R-module $R(n)$ gives rise to a line bundle $\mathscr{O}_{\mathscr{X}}(n)$ on $\mathscr{X}$, although the sheaf $\mathscr{O}_X(n)$ on X may fail to be locally free.

Example 2.1. Let $d_1, \dots, d_n$ be positive integers, and let $k[t_1, \dots, t_n]$ denote the weighted polynomial ring in which t_i has degree d_i. Then the *weighted projective stack* with the given weights is defined as

$$\mathscr{P}(d_1, \dots, d_n) = \mathscr{P}roj(k[t_1, \dots, t_n])$$

and its coarse space is the usual weighted projective space

$$\mathbf{P}(d_1, \dots, d_n) = \operatorname{Proj}(k[t_1, \dots, t_n]).$$

Definition 2.2. Let G be a reductive group acting on a projective scheme $Y \subset \mathbf{P}^N$ via a linear representation. Let S be the homogeneous coordinate ring of Y. Then the stack

$$\mathscr{X} = \mathscr{P}roj(S^G)$$

is the *stacky GIT quotient* of the linearized action of G on Y.

If f is a homogeneous element in $R = S^G$ of degree $r \neq 0$, the ring $R/(f-1)$ is $\mathbf{Z}/(r)$-graded, and there is a corresponding action of the cyclic group $\boldsymbol{\mu}_r = \operatorname{Spec} k[t]/(t^r - 1)$ on its spectrum. The stack quotient $[\operatorname{Spec}(R/(f-1))/\boldsymbol{\mu}_r]$ is an open substack of $\mathscr{P}roj(R)$, and for f running through a generator set of R, these open substacks form an open cover. Thus the stacky GIT quotient is a DM stack with cyclic automorphism groups.

3. Root stacks

We fix a DM stack $\mathscr{X}$, a line bundle $\mathscr{L}$ on $\mathscr{X}$ with a global section s, and a natural number r. Associated to these data, there is a canonically defined stack

$$\pi\colon \mathscr{X}[\sqrt[r]{s}] \to \mathscr{X}$$

over $\mathscr{X}$, called the r'th root along s.

Briefly, $\mathscr{X}[\sqrt[r]{s}]$ is obtained from $\mathscr{X}$ by adding $\boldsymbol{\mu}_r$ to the automorphism groups along the vanishing locus of s, enabling one to extract an r'th root of $\pi^*(s)$. Away from the vanishing locus of s, the map π is an isomorphism.

More precisely, an object of $\mathscr{X}[\sqrt[r]{s}]$ over a scheme T consists of a map $f\colon T \to \mathscr{X}$, together with a line bundle $\mathscr{M}$ on T with a global section t, and an isomorphism

$$\mathscr{M}^r \cong f^*(\mathscr{L})$$

sending t^r to s. The foundations of this construction can be found in a paper by Cadman [5]. In particular, Cadman shows that the root construction applied to a Deligne-Mumford stack is again Deligne-Mumford.

Example 3.1. Let $X = \operatorname{Spec}(R)$ be an affine scheme, and let $s \in R$, considered as a section of the trivial line bundle. Then the r'th root stack along s is the stack quotient

$$X[\sqrt[r]{s}] = [\operatorname{Spec}\left(R[t]/(t^r - s)\right)/\boldsymbol{\mu}_r]$$

where the $\boldsymbol{\mu}_r$-action corresponds to the canonical $\mathbf{Z}/(r)$-grading of $R[t]/(t^r - s)$.

Our aim is to establish a graded analogue of this example. To state the result, we introduce the following notation: If $R = \bigoplus_{d\geq 0} R_d$ is a graded ring and n is a natural number, let $R^{(1/n)}$ be the same ring with grading defined by declaring that $d \in R_n$ has degree dn in $R^{(1/n)}$. Note that $R^{(1/n)}_d = 0$ unless n divides d. In the following we use the notation $\mathscr{P}roj(R)$ for the stack (2.1).

Lemma 3.2. *Let $R = \bigoplus_{n\geq 0} R_n$ be a graded k-algebra with $R_0 = k$, and let $\mathscr{X} = \mathscr{P}roj(R)$. Let $s \in R_n$ be a homogeneous element, considered as a global section of $\mathscr{O}_{\mathscr{X}}(n)$. Let r be a natural number, and assume that r and n have no common factors. Then the r'th root stack of $\mathscr{X}$ along s is*

$$\mathscr{X}[\sqrt[r]{s}] = \mathscr{P}roj(S) \qquad \textit{where} \qquad S = R^{(1/n)}[t]/(t^r - s)$$

with grading defined by letting t have degree n.

The lemma fails without the condition that r and n have no common factors, as the following example shows.

Example 3.3. Let $R = k[x_0, \dots, x_n]$ where the x_i's have degree 1. Then $\mathscr{X}$ is the scheme $\mathbf{P}^n$. Consider the square root stack along a quadratic hypersurface, so let $s \in R$ have degree 2. A point on the square root stack has automorphism group $\boldsymbol{\mu}_2$ if it belongs to the vanishing locus of s; otherwise its automorphism group is trivial. On the other hand

$$S = k[x_0, \dots, x_n, t]/(t^2 - s).$$

The grading defined in the lemma is such that the x_i's and t all have degree 2. But then $\mathscr{P}roj(S)$ has $\boldsymbol{\mu}_2$ as automorphism group everywhere, and is thus not the square root stack along s. The only other sensible grading on S is that in which x_i and t have degree 1, but then $\mathscr{P}roj(S)$ would be a scheme, and we still do not get the square root stack.

Proof of Lemma 3.2. Let $X = \operatorname{Spec}(R) \setminus \{0\}$ and $Y = \operatorname{Spec}(S) \setminus \{0\}$, equipped with $\mathbf{G}_m$-actions

$$\sigma_X\colon \mathbf{G}_m \times X \to X$$
$$\sigma_Y\colon \mathbf{G}_m \times Y \to Y.$$

Viewing Y as the subscheme of $X \times \mathbf{A}^1$ defined by $t^r = s$, the action σ_Y is the restriction of the action on $X \times \mathbf{A}^1$ given on T-valued points by

$$(x, a) \mapsto (\sigma_X(\xi^r, x), \xi^n a) \tag{3.1}$$

for $(x, a) \in X(T) \times \mathbf{A}^1(T)$ and $\xi \in \mathbf{G}_m(T)$.

The claim is that $\mathscr{Y} = [Y/\mathbf{G}_m]$ is the r'th root stack of $\mathscr{X} = [X/\mathbf{G}_m]$ along s. We will show how to map objects in $\mathscr{X}[\sqrt[r]{s}]$ over a scheme T to objects in $\mathscr{Y}$ over T and conversely, leaving out the straight forward verification that these maps are quasi-inverse functors in a natural way. With reference to the diagram

$$\begin{tikzcd}
Q \arrow[r, "{(g,u)}"] \arrow[dr, "\pi"] \arrow[dd, "q"'] & X \times \mathbf{A}^1 \arrow[dr] & \\
& P \arrow[r, "f"] \arrow[dl, "p"] & X \\
T & &
\end{tikzcd} \tag{3.2}$$

the objects in question are given by the following data:

(1) An object in $\mathscr{Y}$ over S is a $\mathbf{G}_m$-torsor $q\colon Q \to T$ together with a $\mathbf{G}_m$-equivariant map $Q \to Y$. Viewing Y as a subscheme of $X \times \mathbf{A}^1$, the latter becomes a pair (g, u) as in the upper part of diagram (3.2), which is equivariant with respect to the action (3.1) on the target.

(2) An object in $\mathscr{X}[\sqrt[r]{s}]$ over T is a $\mathbf{G}_m$-torsor $p\colon P \to T$ together with a $\mathbf{G}_m$-equivariant map f as in the lower part of diagram (3.2), a $\mathbf{G}_m$-linearized line bundle L over P with an equivariant section $v \in \Gamma^{\mathbf{G}_m}(P, L)$ and a $\mathbf{G}_m$-equivariant isomorphism

$$L^r \cong P \times \mathbf{A}^1_{(n)} \quad (= f^*(X \times \mathbf{A}^1_{(n)}))$$

which identifies v^r with $f^*(s)$. Here we write $\mathbf{A}^1_{(n)}$ for the affine line equipped with the $\mathbf{G}_m$-action of weight n.

Given data (1), let $P = Q/\boldsymbol{\mu}_r$ and let $p\colon P \to T$ be the map induced by q. This is a $\mathbf{G}_m$-torsor with respect to the induced action of $\mathbf{G}_m/\boldsymbol{\mu}_r \cong \mathbf{G}_m$. Moreover, g induces an equivariant map f making diagram (3.2) commute. On P there is the $\mathbf{G}_m$-linearized line bundle

$$L = (Q \times \mathbf{A}^1_{(n)})/\boldsymbol{\mu}_r$$

with a section $v \in \Gamma^{\mathbf{G}_m}(P, L)$ induced by the unity section of $Q \times \mathbf{A}^1_{(n)}$, and a canonical trivialization

$$L^r \cong (Q \times \mathbf{A}^1_{(n)})^r/\boldsymbol{\mu}_r \cong P \times \mathbf{A}^1_{(n)}.$$

This defines data as in (2).

Conversely, let data (2) be given. The line bundle L with the trivialization of L^r gives rise to a $\boldsymbol{\mu}_r$-torsor $\pi\colon Q \to P$, defined as follows: Identifying $(L^\vee)^r$ with the trivial line bundle, we let $Q \subset L^\vee$ be the r'th roots of unity in each fibre. This is clearly a $\boldsymbol{\mu}_r$-torsor under the action of multiplication in the fibres. Now we define a new $\boldsymbol{\mu}_r$-action on Q by letting $\xi \in \boldsymbol{\mu}_r$ act by multiplication with ξ^n in the fibres. Since r and n are relatively prime, the n'th power endomorphism on $\mathbf{G}_m$ induces an automorphism on $\boldsymbol{\mu}_r$, so Q is a $\boldsymbol{\mu}_r$-torsor also under this new action. Moreover, it extends to a $\mathbf{G}_m$-action as follows:

Let $\xi \in \mathbf{G}_m$ act on $L^\vee$ by composing the contragradient action of ξ^r (using the given $\mathbf{G}_m$-action on L) with multiplication by ξ^n in the fibres:

$$\begin{tikzcd}
\mathbf{G}_m \times L^\vee \arrow[r, "(n,r)\times 1_{L^\vee}"] \arrow[ddr] & \mathbf{G}_m \times \mathbf{G}_m \times L^\vee \arrow[d, "1_{\mathbf{G}_m} \times \text{(contragrad.)}"] \\
& \mathbf{G}_m \times L^\vee \arrow[d, "\text{(scalar mult.)}"] \\
& L^\vee
\end{tikzcd}$$

Then $Q \subset L^\vee$ is $\mathbf{G}_m$-invariant, and the induced action extends the $\boldsymbol{\mu}_r$-action defined above. The given $\mathbf{G}_m$-action on P agrees with the induced action of $\mathbf{G}_m/\boldsymbol{\mu}_r \cong \mathbf{G}_m$ on $Q/\boldsymbol{\mu}_r \cong P$, and it follows that Q is a $\mathbf{G}_m$-torsor over T. We now let $g = f \circ \pi$ and let u be the restriction of

$$v^\vee \colon L^\vee \to \mathbf{A}^1$$

to Q. This defines data as in (1). □

4. Rigidification

Intuitively, the rigidification of a given stack is "the same stack" with the general automorphism group removed.

Definition 4.1. The *rigidification* of an irreducible DM stack $\mathscr{X}$ is a dominant map

$$f \colon \mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$$

to another DM stack $\mathscr{X}^{\mathrm{rig}}$, such that the automorphism group $\mathrm{Aut}(x)$ of a general point $x \in \mathscr{X}^{\mathrm{rig}}$ is trivial, and such that f is universal with this property.

We spell out the meaning of universality: For every dominant map $g \colon \mathscr{X} \to \mathscr{Y}$, to a DM stack $\mathscr{Y}$ whose general points have trivial automorphism groups, we require the existence of a map $h \colon \mathscr{X}^{\mathrm{rig}} \to \mathscr{Y}$ making the diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[d, "f"'] \arrow[dr, "g"] & \\
\mathscr{X}^{\mathrm{rig}} \arrow[r, "h"] & \mathscr{Y}
\end{tikzcd}$$

2-commutative, and the map h is unique up to unique natural equivalence.

Remark 4.2. Rigidifications are defined in the literature in greater generality [1, 2]. Namely, one chooses a subgroup stack G of the inertia stack $I(\mathscr{X})$, and defines the rigidification with respect to G to be a stack receiving a map from $\mathscr{X}$, with automorphisms belonging to G being killed, and universal with this property. The rigidification of Definition 4.1 is the special case where G is taken to be the closure of the union of the automorphism groups of general points x of $\mathscr{X}$. The rigidification in this sense is known to exist [2, Example A.3] under general conditions. In order to keep the presentation self contained we give a direct construction in Proposition 4.3 in the situation we need here.

Without any conditions on $\mathscr{X}$, the rigidification does not necessarily exist. We treat an easy special case where it does exist: Consider a DM stack of the form $[X/G]$ for an algebraic group G acting on an irreducible scheme X, and suppose that the general stabilizer group equals the common stabilizer group

$$H = \{g \in G \,|\, gx = x \;\forall\, x \in X\}\,.$$

More precisely, we assume that there is an open subset $U \subseteq X$ such that $G_x = H$ for all $x \in U$. Note that H is normal, and the induced action of $G' = G/H$ has generically trivial stabilizer.

Proposition 4.3. *Let G be an algebraic group scheme acting on an irreducible scheme X, such that $[X/G]$ is a DM stack and such that the general stabilizer group equals the common stabilizer group $H \subseteq G$. Then the stack quotient $[X/G]$ admits a rigidification, in fact*

$$[X/G]^{\mathrm{rig}} \cong [X/G']$$

where $G' = G/H$.

Before proving the proposition, we establish a lemma.

Lemma 4.4. *Let $\mathscr{Y}$ be a DM stack containing an algebraic space $U \subseteq \mathscr{Y}$ as an open dense substack. Let*

$$\alpha, \beta \colon S \rightrightarrows \mathscr{Y}$$

be two maps from a scheme S, such that the restrictions of α and β to every component of S are dominant. Then equivalence $\alpha \cong \beta$ is a local property on S. Precisely, if $p \colon T \to S$ is a surjective, flat map, locally of finite presentation, such that $p^*(\alpha)$ and $p^*(\beta)$ are equivalent, then already α and β are equivalent.

Proof. There is an open dense subset $S' \subset S$ whose image under both α and β is contained in U. Since the restricted maps $\alpha|_{S'}$ and $\beta|_{S'}$ have the algebraic space U as codomain, it is clear that equivalence $\alpha|_{S'} \cong \beta|_{S'}$ is a local property on S', and so holds by assumption.

Recall that, according to our conventions, the DM stack $\mathscr{Y}$ is separated. Thus there is a closed subscheme $S'' \subseteq S$ which is universal with the property that the restrictions of α and β to S'' are equivalent. We have established that S'' contains S', which is dense in S. Thus $S'' = S$ and we conclude that α and β are equivalent. $\square$

Proof of the proposition. Clearly, the quotient stack $[X/G']$ has generically trivial stabilizer, and admits a canonical surjective map

$$f \colon [X/G] \to [X/G'].$$

Let $g \colon [X/G] \to \mathscr{Y}$ be another dominant map to a DM stack $\mathscr{Y}$ with generically trivial automorphism groups, and consider the diagram

$$\begin{tikzcd}
 & & X \arrow[dr, "\pi"] & & \\
G \times X \arrow[r] \arrow[urr, "\sigma"] \arrow[drr, "p_2"'] & G' \times X \arrow[ur, "\sigma'"'] \arrow[dr, "p_2"] & & {[X/G]} \arrow[r, "g"] & \mathscr{Y} \\
 & & X \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

where σ and σ' are the actions, p_2 is second projection and π is the quotient map. We claim that $g \circ \pi \circ \sigma'$ and $g \circ \pi \circ p_2$ are equivalent maps $G' \times X \to \mathscr{Y}$. Once this is established, it follows from the universality of the quotient stack $[X/G']$ that g factors through f as required.

We apply the lemma as follows: Let $U \subset \mathscr{Y}$ be an open dense substack with trivial automorphism groups everywhere. It is an algebraic space. By definition of the quotient $[X/G]$, the two maps $\pi \circ \sigma$ and $\pi \circ p_2$ are equivalent. Hence also their compositions with g are equivalent. Taking $R \to S$ in the lemma to be the flat surjective map $G \times X \to G' \times X$, we find that the two maps from $G' \times X$ to $\mathscr{Y}$ are already equivalent, as claimed. $\square$

Corollary 4.5. *Let $R = \bigoplus_{d\geq 0} R_d$ be a graded k-algebra with $R_0 = k$, whose nilradical is prime (i.e. $\operatorname{Spec}(R)$ is irreducible). Then $\mathscr{P}roj(R)$ admits a rigidification. In fact, letting*

$$n = \operatorname{hcf}\{d \,|\, R_d \neq 0\},$$

we have

$$\mathscr{P}roj(R)^{\mathrm{rig}} \cong \mathscr{P}roj(R^{(n)}).$$

Proof. With n as above, the subgroup $\boldsymbol{\mu}_n \subset \mathbf{G}_m$ acts trivially on $\operatorname{Spec}(R)$. On the other hand, there exists a finite set of homogeneous elements $f_1, \dots, f_r$ in R such that the highest common factor of their degrees equals n. Let $U \subset \operatorname{Spec}(R)$ be the open set defined by the simultaneous nonvanishing of the f_i's. Then the stabilizer group of any point in U is exactly $\boldsymbol{\mu}_n$. Thus $\boldsymbol{\mu}_n$ is both the common stabilizer group and the generic stabilizer group, so the proposition applies. The quotient $\mathbf{G}_m/\boldsymbol{\mu}_n$ is again isomorphic to $\mathbf{G}_m$, and the induced action corresponds to the grading in which $f \in R_d$ is given degree d/n. This is by definition the grading on $R^{(n)}$. □

Remark 4.6. In the last Corollary, $R^{(n)}$ and R are essentially the same ring, only with a different grading. Geometrically, this can be phrased as follows: The stack $\mathscr{P}roj(R)$ is the n'th root stack of $\mathscr{O}(1)$ on $\mathscr{P}roj(R^{(n)})$, defined similarly as the root stack in Section 3, only without the section s (this construction can also be found in Cadman's paper [5]). At the level of points with automorphisms groups, $\mathscr{P}roj(R)$ is obtained from $\mathscr{P}roj(R^{(n)})$ by sticking in an extra automorphism group $\boldsymbol{\mu}_n$ everywhere. More precisely,

$$\mathscr{P}roj(R) \to \mathscr{P}roj(R^{(n)})$$

is an *essentially trivial $\boldsymbol{\mu}_n$-gerbe*. We refer the reader to Lieblich [13] and Fantechi et. al. [9] for systematic expositions, but mention briefly that a $\boldsymbol{\mu}_n$-gerbe over a stack $\mathscr{X}$ corresponds to an element of $H^2(\mathscr{X}, \boldsymbol{\mu}_n)$, and is called essentially trivial if the push forward to $H^2(\mathscr{X}, \mathbf{G}_m)$ vanishes. This is equivalent [13, Proposition 2.3.4.4] [9, Remark 6.4] to the statement that the gerbe is the n'th root stack of a line bundle on $\mathscr{X}$.

5. The stacky GIT quotient

The invariant ring of binary quartics, and that of cubic plane curves, are weighted polynomial rings in two variables (see Sections 7.1 and 8.1). In this section we study the more interesting invariant rings for binary quintics, binary sextics, and cubic surfaces: All of these have the structure (see Sections 7.2, 7.3 and 8.2)

$$R \cong k[t_1, \dots, t_{n+1}]/(t_{n+1}^2 - F(t_1, \dots, t_n)) \tag{5.1}$$

where t_i are homogeneous generators of some positive weight d_i, and F is a weighted homogeneous polynomial of degree $2d_{n+1}$. The weights fulfil the following three conditions:

(i) The highest common factor d of $d_1, \dots, d_n$ does not divide d_{n+1}.
(ii) The weights $e_i = d_i/d$, for $i \leq n$, are well formed, i.e. no $n-1$ among them have a common factor.
(iii) $2d_{n+1}/d$ is even.

The first condition says that t_{n+1} is not an element of the subring $R^{(d)} \subseteq R$, which thus is a weighted polynomial ring. The second condition says that its generators $t_1, \dots, t_n$ have well formed weights. The last condition says that the degree of F, as an element of $R^{(d)}$, is even, which is the condition needed to apply Lemma 3.2 to extract a square root.

Recall [9, Section 4.1] that any variety X, with at worst finite quotient singularities, is in a canonical way the coarse space of a smooth DM stack, the *canonical stack* X^{can}.

More precisely, for any smooth DM stack $\mathscr{Y}$ with X as coarse space, there is a unique map $\mathscr{Y} \to X^{\mathrm{can}}$, compatible with the maps to X. Thus, if $\mathscr{X}$ is a DM stack admitting a smooth rigidification, having a variety X with finite quotient singularities as coarse space, then the universal properties of the rigidification and the canonical stack yield a factorization of the canonical map $\mathscr{X} \to X$ as

$$\mathscr{X} \to \mathscr{X}^{\mathrm{rig}} \to X^{\mathrm{can}} \to X.$$

Theorem 5.1. *Let R be a graded k-algebra of the form* (5.1)*, satisfying conditions (i), (ii) and (iii) above. Let $\mathscr{X}$ be the stack $\mathscr{P}roj(R)$ and let X be its coarse space* $\mathrm{Proj}(R)$*.*

(1) *X is the weighted projective space $\mathbf{P}(e_1, \dots, e_n)$, and its canonical stack is the weighted projective stack $X^{\mathrm{can}} = \mathscr{P}(e_1, \dots, e_n) = \mathscr{P}roj(R^{(d)})$.*
(2) *The rigidification of $\mathscr{X}$ is $\mathscr{X}^{\mathrm{rig}} = \mathscr{P}roj(R^{(d/2)})$.*
(3) *The map $\mathscr{X}^{\mathrm{rig}} \to X^{\mathrm{can}}$ is the square root along F, considered as a section of $\mathscr{O}_{X^{\mathrm{can}}}(2d_{n+1}/d)$.*

Proof. The coarse moduli space of the stack $\mathscr{P}roj(R)$ is the scheme $\mathrm{Proj}(R)$. Since $\mathrm{Proj}(R) \cong \mathrm{Proj}(R^{(d)})$, and

$$R^{(d)} \cong k[t_1, \dots, t_n]$$

is a weighted polynomial ring, where each generator t_i has degree $e_i = d_i/d$, it is clear that the coarse moduli space is the weighted projective space as claimed. It is a standard fact [9, Example 7.25] that its canonical smooth stack is $\mathscr{P}(e_1, \dots, e_n) = \mathscr{P}roj(R^{(d)})$, using that the weights are well formed. This proves (1).

Next we apply Corollary 4.5. Since $2d_{n+1} = \deg F$ and d divides $\deg F$, we see that d is even and the highest common factor of $d_1, \dots, d_{n+1}$ is $d/2$ (using the assumption that d does not divide d_{n+1}). This proves (2).

Finally, Lemma 3.2 immediately gives $\mathscr{P}roj(R^{(d/2)})$ as the square root stack of $\mathscr{P}roj(R^{(d)})$ along F. This proves (3). □

Remark 5.2. The theorem tells us in particular that the stack $\mathscr{X}$ remembers not only its coarse moduli space X, but also the divisor defined by F. Conversely, knowing X and F, we can reconstruct the rigidification of $\mathscr{X}$ by extracting a square root of F on the canonical stack associated to X. Finally, $\mathscr{X}$ is an essentially trivial $\boldsymbol{\mu}_{(d/2)}$-gerbe over its rigidification, as in Remark 4.6.

6. Symmetries of binary forms

The aim of this section is to survey Klein's classification [12] of binary forms according to their symmetries, i.e. their stabilizer groups. Throughout, we identify forms that differ by a nonzero scalar factor. Thus, by the stabilizer group of a binary form f, we shall mean the elements of $\mathrm{SL}(2)$ under which f is semi-invariant, i.e. invariant up to a nonzero scalar factor.

Recall that a binary form of degree d is stable if and only if all its roots have multiplicity strictly less than $d/2$. Such a binary form has finite stabilizer group in $\mathrm{SL}(2)$. More generally, any binary form with at least three distinct zeros has finite stabilizer group. This leaves just the case $f = x^n y^m$ (modulo $\mathrm{SL}(2)$ and scale), whose stabilizer group consists of all diagonal matrices in $\mathrm{SL}(2)$ if $n \neq m$, and all diagonal and antidiagonal matrices if $n = m$. From now on we assume that f is a binary form with finite stabilizer group $G \subset \mathrm{SL}(2)$. Hence G is a cyclic, dihedral, tetrahedral, octahedral or icosahedral group, by the well known classification of finite subgroups of $\mathrm{SL}(2)$. More precisely, a conjugate of G equals one of the subgroups listed in Table 1. The conjugation corresponds to picking

Group	Generators
C_n	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ $\qquad$ ($\epsilon = \sqrt[2n]{1}$ primitive)
D_n	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\qquad$ ($\epsilon = \sqrt[2n]{1}$ primitive)
T	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$
O	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$
I	$\begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon-\epsilon^4 & \epsilon^3-\epsilon^2 \\ \epsilon^3-\epsilon^2 & \epsilon^4-\epsilon \end{pmatrix}$ $\quad$ ($\epsilon = \sqrt[5]{1}$ primitive)

TABLE 1. Polyhedral groups

Group	Ground forms
D_n	$F_1 = x^n + y^n$ $F_2 = x^n - y^n$ $F_3 = xy$
T	$F_1 = x^4 + 2\sqrt{-3}x^2y^2 + y^4$ $F_2 = x^4 - 2\sqrt{-3}x^2y^2 + y^4$ $F_3 = xy(x^4 - y^4)$
O	$F_1 = xy(x^4 - y^4)$ $F_2 = x^8 + 14x^4y^4 + y^8$ $F_3 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$
I	$F_1 = xy(x^{10} + 11x^5y^5 - y^{10})$ $F_2 = -(x^{20} + y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}$ $F_3 = (x^{30} + y^{30}) + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20})$

TABLE 2. Ground forms

another representative for the orbit of f under $\mathrm{SL}(2)$, since $\gamma G \gamma^{-1}$ is the stabilizer group of γf.

Thus, to classify binary forms with finite stabilizer group, it suffices to determine the semi-invariant forms for each group G in Table 1. A general G-orbit in $\mathbf{P}^1$ has degree $|G|/2$. For non-cyclic G, there are precisely three special orbits of smaller degree, defined by the vanishing of three so called ground forms F_1, F_2, and F_3. These are listed in Table 2. We put $\nu_i = |G|/(2\deg F_i)$.

Lemma 6.1 (Klein [12]). *A binary form f is semi-invariant under the cyclic group C_n if and only if (up to a scalar factor)*

$$f = x^\alpha y^\beta \prod_{i=1}^{N} (\lambda_i x^n + \mu_i y^n)$$

where α, β and N are nonnegative integers, and $(\lambda_i : \mu_i) \in \mathbf{P}^1$ are parameters.

A binary form f is semi-invariant under one of the groups D_n, T, O, I if and only if (up to a scalar factor)

$$f = F_1^{\alpha} F_2^{\beta} F_3^{\gamma} \prod_{i=1}^{N} (\lambda_i F_1^{\nu_1} + \mu_i F_2^{\nu_2})$$

where α, β, γ and N are nonnegative integers, $(\lambda_i : \mu_i) \in \mathbf{P}^1$ are parameters, and F_1, F_2 and F_3 are the ground forms associated with the group.

Remark 6.2. Consider the central projection of an inscribed regular polyhedron onto the Riemann sphere $\mathbf{P}^1$ (over $\mathbf{C}$). Then the ground forms of the group corresponding to the polyhedron have as zero loci the vertices, the midpoints of the faces and the midpoints of the edges, respectively. A similar statement applies to the cyclic and dihedral groups. The generic orbits, on the other hand, are the zero loci of the forms $\lambda F_1^{\nu_1} + \mu F_2^{\nu_2}$.

For each fixed (and small) d, we can use the lemma to explicitly write down all semi-invariants of degree d for each group in Table 1. This easily leads to the following classification of binary forms of low degree, where we include only the stable cases, and write $\mathrm{Stab}(f) \subset \mathrm{SL}(2)$ for the stabilizer subgroup of f.

6.1. **Binary quartics.** Every stable, i.e. square free, quartic is equivalent under the $\mathrm{SL}(2)$-action to

$$f = \lambda(x^2 + y^2)^2 + \mu(x^2 - y^2)^2$$

for some $(\lambda : \mu)$, and so has the dihedral group D_2 contained in its stabilizer. Modulo the $\mathrm{SL}(2)$-action, there are exactly two quartics with larger stabilizer group:

(I)	$f = x^4 + y^4$	$\mathrm{Stab}(f) = D_4$
(II)	$f = x^4 + 2\sqrt{-3}x^2y^2 + y^4$	$\mathrm{Stab}(f) = T$

6.2. **Quintics.** A stable quintic is one with at most double roots. All quintics are stabilized by $C_1 = \{\pm 1\}$. Modulo $\mathrm{SL}(2)$, the quintics with larger stabilizer group are the following:

(I)	$f = x(x^2 + y^2)(\lambda x^2 + \mu y^2)$	$\mathrm{Stab}(f) = C_2$
(II)	$f = x^2(x^3 + y^3)$	$\mathrm{Stab}(f) = C_3$
(III)	$f = x(x^4 + y^4)$	$\mathrm{Stab}(f) = C_4$
(IV)	$f = xy(x^3 + y^3)$	$\mathrm{Stab}(f) = D_3$
(V)	$f = x^5 + y^5$	$\mathrm{Stab}(f) = D_5$

Here, the pair $(\lambda : \mu) \in \mathbf{P}^1$ is a parameter assumed to have generic value, so that the listed cases are disjoint.

6.3. **Sextics.** A stable sextic is one with at most double roots. All sextics are stabilized by $C_1 = \{\pm 1\}$. Modulo $\mathrm{SL}(2)$, the sextics with larger stabilizer group are the following:

(I)	$f = (x^2+y^2)\prod_{i=1}^{2}(\lambda_i x^2 + \mu_i y^2)$	$\mathrm{Stab}(f) = C_2$
(II)	$f = x(x^5+y^5)$	$\mathrm{Stab}(f) = C_5$
(III)	$f = xy(\lambda(x^2+y^2)^2 + \mu(x^2-y^2)^2)$	$\mathrm{Stab}(f) = D_2$
(IV)	$f = \lambda(x^3+y^3)^2 + \mu(x^3-y^3)^2$	$\mathrm{Stab}(f) = D_3$
(V)	$f = x^6+y^6$	$\mathrm{Stab}(f) = D_6$
(VI)	$f = xy(x^4-y^4)$	$\mathrm{Stab}(f) = O$
(VII)	$f = x^2y(x^3+y^3)$	$\mathrm{Stab}(f) = C_3$
(VIII)	$f = x^2(x^4+y^4)$	$\mathrm{Stab}(f) = C_4$

Here again the parameters λ, μ, λ_i, μ_i are assumed to have generic values, so that the listed cases are disjoint.

Bolza [4] produced a list of symmetry groups for square free sextics, which is equivalent to the first six items in our list. As we will return to Bolza's work in Section 7.3, we remark that the labels (I)-(VI) we are using agree with Bolza's.

7. Moduli spaces of binary forms

We let
$$R \subset k[a_0, \dots, a_d]$$
be the invariant ring for the $\mathrm{SL}(2)$-action on degree d binary forms
$$f = a_0x^d + a_1x^{d-1}y + \dots a_dy^d.$$
In this section we apply the results from the previous sections to analyse the geometry of $\mathscr{P}roj(R)$, for small values of d. In particular, we describe the loci of binary forms with prescribed symmetry group in terms of the intrinsic geometry of the stacky GIT quotient.

For $d = 4$, we find that quartics with extra symmetries show up as points in the stacky GIT quotient with nontrivial automorphism groups.

For $d = 5$ and $d = 6$, the invariant ring R has the form studied in Section 5. Thus, from the stacky GIT quotient we obtain the usual GIT quotient scheme $\mathrm{Proj}(R)$ together with the divisor $Z(F)$. We find that the binary forms corresponding to singularities of the scheme $\mathrm{Proj}(R)$ have special symmetry groups, but there are also loci of binary forms with symmetries that do not give rise to singularities. However, these loci show up as $Z(F)$, its singularities, or the singularities of its singular locus. Thus the knowledge of $\mathrm{Proj}(R)$ together with the divisor $Z(F)$ suffices to enable a geometric description of all loci of binary forms with prescribed symmetry group.

For the explicit computations needed in this section we rely on a computer algebra system such as Singular [10]. Armed with such a system, the calculations are straight forward, and we only give the results.

7.1. **Binary quartics.** The invariant ring R for binary forms of degree 4 is freely generated by two homogeneous invariants
$$I_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$I_3 = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3,$$

where the subscript i of an invariant I_i indicates its degree. Thus $R = k[I_2, I_3]$ is a weighted polynomial ring, and the GIT quotient is

$$X = \mathrm{Proj}(R) \cong \mathbf{P}^1.$$

In particular it is a homogeneous space, so, morally, it looks the same at all points. Thus the geometry of the quotient does not single out the two points corresponding to the special quartics (I) and (II) from Section 6.1. On the other hand, the stacky GIT quotient

$$\mathscr{X} = \mathscr{P}roj(R) = \mathscr{P}(2,3)$$

is a weighted projective line, in the stack sense, and the two special points $(1 : 0)$ and $(0 : 1)$ corresponding to special quartics are distinguished by having automorphism groups $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_3$, respectively.

In the language of the root construction, the stack $\mathscr{X}$ is obtained from its coarse space X by extracting a square root of $(1 : 0)$ and a cube root of $(0 : 1)$.

7.2. **Binary quintics.** The invariant ring for binary quintics can be written

$$R = k[I_4, I_8, I_{12}, I_{18}]/(I_{18}^2 - F(I_4, I_8, I_{12}))$$

where the generators I_i are homogeneous of degree i and F is (weighted) homogeneous of degree 36.

Thus Theorem 5.1 applies: The GIT quotient scheme is the weighted projective plane

$$X = \mathrm{Proj}(R) = \mathbf{P}(1,2,3)$$

and the canonical stack X^{can} is the weighted projective stack $\mathscr{P}(1,2,3)$. The stacky GIT quotient $\mathscr{X} = \mathscr{P}roj(R)$ is an essentially trivial $\boldsymbol{\mu}_2$-gerbe over its rigidification $\mathscr{X}^{\mathrm{rig}}$, which is obtained from X^{can} by extracting a square root of F.

We note that the weighted projective plane X has cyclic quotient singularities at $(0 : 1 : 0)$ and $(0 : 0 : 1)$, and is otherwise smooth. The stack $\mathscr{X}$ also remembers the divisor defined by F, which we now analyse.

The generators I_i for the invariant ring are not uniquely defined. In the following we choose the generators given by Schur [14]. With this choice, we have[1]

$$324F(I_4, I_8, I_{12}) = -9I_4I_8^4 - 24I_8^3I_{12} + 6I_4^2I_8^2I_{12} + 72I_4I_8I_{12}^2 + 144I_{12}^3 - I_4^3I_{12}^2, \tag{7.1}$$

which is irreducible and is singular at $(1 : 0 : 0)$ and $(-3 : 3 : 3)$. Denoting the closures of the loci of special quintics with Roman numerals (I)-(V), according to the list in Section 6.2, we have:

- (I) is the divisor $Z(F)$
- (II) and (III) are the two singularities of $\mathbf{P}(1,2,3)$
- (IV) and (VI) are the two singularities of $Z(F)$

Moreover, the curve $Z(F)$ passes through (III) but avoids (II). The situation is summarized in Figure 1.

[1]Schur does not give the relation, but it can be found in Elliot's book [8]. Elliot's and Schur's invariants I_i agree up to scale.

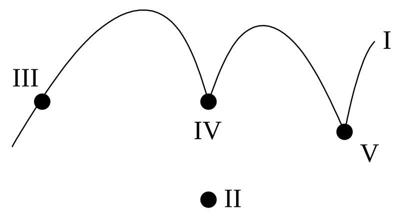

FIGURE 1. Inside the moduli space of binary quintics.

7.3. **Binary sextics.** The invariant ring for binary sextics can be written

$$R = k[I_2, I_4, I_6, I_{10}, I_{15}]/(I_{15}^2 - F(I_2, I_4, I_6, I_{10}))$$

where the generators I_i are homogeneous of degree i and F is (weighted) homogeneous of degree 30.

Thus Theorem 5.1 applies: The GIT quotient scheme is the weighted projective space

$$X = \operatorname{Proj}(R) = \mathbf{P}(1, 2, 3, 5)$$

and the canonical stack X^{can} is the weighted projective stack $\mathscr{P}(1, 2, 3, 5)$. The stacky GIT quotient $\mathscr{X} = \mathscr{P}roj(R)$ is its own rigidification, and it is obtained from X^{can} by extracting a square root of F.

The weighted projective space X has cyclic quotient singularities at $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$ and is otherwise smooth. We next analyse the divisor $Z(F)$.

For a specific choice of generators I_i, Clebsch [6] gives F explicitly as twice the determinant of the 3×3 symmetric matrix (a_{ij}) with entries

$$\begin{aligned}
a_{11} &= 2I_6 + \tfrac{1}{3}I_2I_4 & a_{23} &= \tfrac{1}{3}I_4(I_4^2 + I_2I_6) + \tfrac{1}{3}I_6(2I_6 + \tfrac{1}{3}I_2I_4)\\
a_{12} &= \tfrac{2}{3}(I_4^2 + I_2I_6) & a_{33} &= \tfrac{1}{2}I_6I_{10} + \tfrac{2}{9}I_6(I_4^2 + I_2I_6)\\
a_{13} &= I_{10} & a_{22} &= I_{10}.
\end{aligned}$$

This polynomial F is irreducible. Its zero locus $Z(F)$ is a surface which is singular along a curve, having two components. Each component has one singular point.

It turns out that these loci matches the classification of sextics from Section 6.3: Again we use roman numerals (I)-(VIII) for the closures of the loci in X corresponding to the special sextics. Equations for the loci (I)-(VI) were determined by Bolza [4], and the points corresponding to the remaining special sextics (VII) and (VIII) can be determined by evaluating explicit expressions for the invariants I_i. The results are as follows:

- (I) is the divisor $Z(F)$
- (II), (VII) and (VIII) are the three singularities of $\mathbf{P}(1, 2, 3, 5)$
- (III) and (IV) are the two curves along which $Z(F)$ is singular
- (V) is the singular point of the curve (III)
- (VI) is the singular point of the curve (IV)

Furthermore, the curves (III) and (IV) intersect in (V), (VI) and one additional point. The latter corresponds to strictly semistable sextics, i.e. sextics with a triple root.

One also checks that the surface $Z(F)$ does not contain (II) and (VII), but it contains (VIII) and is smooth there. The situation is summarized in Figure 2.

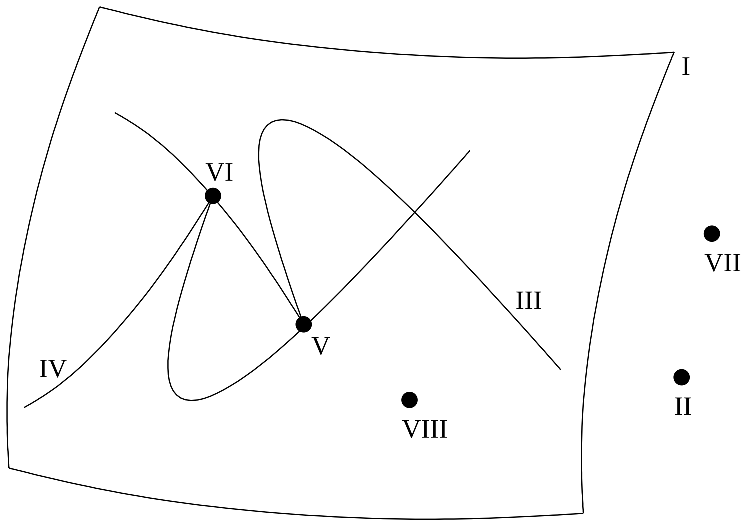

FIGURE 2. Inside the moduli space of binary sextics.

8. CUBIC CURVES AND SURFACES

In this section we describe the relation between the GIT quotient scheme and the stacky GIT quotient corresponding to cubic plane curves and cubic surfaces in space.

8.1. **Cubic plane curves.** The invariant ring R for the action of $\mathrm{SL}(3)$ on cubic forms in three variables can be written

$$R = k[I_4, I_6]$$

where the generators I_i are homogeneous degree i polynomials in the coefficients of the cubic form. Thus the GIT quotient scheme is $\mathbf{P}^1$. By Corollary 4.5, the stacky GIT quotient $\mathscr{P}roj(R)$ is an essentially trivial $\boldsymbol{\mu}_2$-gerbe over its rigidification $\mathscr{P}roj(R^{(2)})$, which is the weighted projective stack $\mathscr{P}(2,3)$. As in Section 7.1, this stack is obtained from its coarse space $\mathbf{P}^1$ by extracting a square root of $(0:1)$ and a cube root of $(1:0)$.

8.2. **Cubic surfaces.** The invariant ring R for the action of $\mathrm{SL}(4)$ on cubic forms in four variables can be written

$$R = k[I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}]/(I_{100}^2 - F(I_8, I_{16}, I_{24}, I_{32}, I_{40}))$$

where the generators I_i are homogeneous of degree i and F is (weighted) homogeneous of degree 200.

Thus Theorem 5.1 applies: The GIT quotient scheme is the weighted projective space

$$X = \mathrm{Proj}(R) = \mathbf{P}(1,2,3,4,5)$$

and the canonical stack X^{can} is the weighted projective stack $\mathscr{P}(1,2,3,4,5)$. The stacky GIT quotient $\mathscr{X} = \mathscr{P}roj(R)$ is an essentially trivial $\boldsymbol{\mu}_4$-gerbe over its rigidification $\mathscr{X}^{\mathrm{rig}}$, which is obtained from X^{can} by extracting a square root of F.

One may expect that the singularities of X, $Z(F)$, the singularities of their singular loci etc., reflect a classification of cubic surfaces according to their symmetries, as was the case for binary forms. We have not investigated this further.

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